

EQUIVARIANT UNITARY BORDISM AND EQUIVARIANT COHOMOLOGY CHERN NUMBERS

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ABSTRACT. By using the universal toric genus and the Kronecker pairing of bordism and cobordism, this paper shows that the integral equivariant cohomology Chern numbers completely determine the equivariant geometric unitary bordism classes of closed unitary G -manifolds, which gives an affirmative answer to the conjecture posed by Guillemin–Ginzburg–Karshon in [18, Remark H.5, §3, Appendix H], where G is a torus. Our approach heavily exploits Quillen’s geometric interpretation of homotopic unitary cobordism theory. As a further application, we also obtain a satisfactory solution of [18, Question (A), §1.1, Appendix H] on unitary Hamiltonian G -manifolds. In particular, our approach can also be applied to the study of $(\mathbb{Z}_2)^k$ -equivariant unoriented bordism, and without the use of Boardman map, it can still work out the classical result of tom Dieck, which states that the $(\mathbb{Z}_2)^k$ -equivariant unoriented bordism class of a smooth closed $(\mathbb{Z}_2)^k$ -manifold is determined by its $(\mathbb{Z}_2)^k$ -equivariant Stiefel–Whitney numbers.

In addition, this paper also shows the equivalence of integral equivariant cohomology Chern numbers and equivariant K-theoretic Chern numbers for determining the equivariant unitary bordism classes of closed unitary G -manifolds by using the developed equivariant Riemann–Roch relation of Atiyah–Hirzebruch type, which implies that, in a different way, we may induce another classical result of tom Dieck, saying that equivariant K-theoretic Chern numbers completely determine the equivariant geometric unitary bordism classes of closed unitary G -manifolds.

1. INTRODUCTION AND MAIN RESULTS

1.1. Unitary bordism. Let Ω_*^U denote the ring formed by the unitary bordism classes of all closed unitary manifolds, where a *unitary manifold* is an oriented smooth compact manifold whose tangent bundle admits a stable almost complex structure. It is known well (cf [25, 26]) that unitary bordism is determined by integral cohomology Chern numbers. Namely, $\alpha \in \Omega_*^U$ is zero if and only if all integral cohomology Chern numbers of α vanish. On the other hand, it is well-known that all possible relations among these integral cohomology Chern numbers are indicated by the Riemann–Roch theorem of Atiyah–Hirzebruch type by relating complex K-theory to rational cohomology (see [5, 20, 28]). This actually implies the equivalence of integral cohomology Chern numbers and K-theoretic Chern numbers for determining the unitary bordism class of a closed unitary manifold. We will give a simple proof (see Proposition 3.5). Therefore, one has that

Theorem 1.1. *In unitary bordism theory there are three equivalent statements as follows:*

- (1) $\alpha \in \Omega_*^U$ is zero.

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- (2) *All integral cohomology Chern numbers of α vanish.*
- (3) *All K-theoretic Chern numbers of α vanish.*

1.2. Equivariant unitary (co)bordism. In the early 1960s, Conner and Floyd ([12, 13, 14]) began the study of geometric equivariant unoriented, oriented and complex bordism theories for smooth closed manifolds with periodic diffeomorphisms, and the subject has also continued to develop and to flourish by extending their ideas to other equivariant (co)bordisms since then. For example, the homotopy theoretic analogue was described by tom Dieck in [29]. Similarly to the nonequivariant case, there are mainly two kinds of equivariant (co)bordism theories: equivariant geometric (co)bordism and equivariant homotopic (co)bordism. However, unlike the nonequivariant case, these two kinds of equivariant (co)bordism theories have quite differences.

In this paper, we are mainly concerned with the case of unitary G -manifolds, where G is a compact Lie group and a *unitary G -manifold* M is a unitary manifold equipped with an effective G -action preserving the tangential stably almost complex structure of M . Two closed unitary G -manifolds M_1, M_2 are said to be *equivariantly geometrically bordant* provided that there exists a compact unitary G -manifold W such that the induced unitary G -structure on the boundary ∂W is equivalent to the unitary G -structure on $M_1 \amalg -M_2$. Similarly to the nonequivariant case, the equivariant geometric bordism classes of closed unitary G -manifolds form a ring, which is called the *equivariant geometric unitary bordism ring* and is denoted by $\Omega_*^{U,G}$.

In his series of papers ([29, 30, 31]), by combining the geometric approach of Conner–Floyd ([13, 14]) and the K-theory approach developed by Atiyah, Bott, Segal and Singer ([4, 7, 8]), tom Dieck developed cobordism analogue of K-theory integrality theorems, and showed that when G is a product of a torus and a finite cyclic group, equivariant K-theoretic Chern numbers determine the equivariant geometric unitary bordism classes of closed unitary G -manifolds, which is stated as follows:

Theorem 1.2 (tom Dieck). *Let G be a product of a torus and a finite cyclic group. Then $\beta \in \Omega_*^{U,G}$ is zero if and only if all equivariant K-theoretic Chern numbers of β vanish.*

In their book [18, Appendix H], Guillemin–Ginzburg–Karshon discussed the problem of calculating the ring \mathcal{H}_*^G of equivariant Hamiltonian bordism classes of all unitary Hamiltonian G -manifolds with integral equivariant cohomology classes $\frac{1}{2\pi}[\omega - \Phi]$, where G is a torus. With respect to the determination of the ring \mathcal{H}_*^G , they designed three series of questions, the first one of which is stated as follows:

Question 1.3 ([18, Question (A), §1.1, Appendix H]). *Do mixed equivariant characteristic numbers form a full system of invariants of Hamiltonian bordism?*

On this question, Guillemin–Ginzburg–Karshon constructed a monomorphism

$$\Sigma_G : \mathcal{H}_*^G \longrightarrow \Omega_{*+2}^{U,G},$$

so that Question 1.3 is equivalent to asking if the integral equivariant cohomology Chern numbers can determine the equivariant geometric unitary bordism for the ring $\Omega_*^{U,G}$. They showed that a closed unitary G -manifold M with only isolated fixed-points represents the zero element in $\Omega_*^{U,G}$ if and only if all integral equivariant cohomology Chern numbers of

M vanish, which gives a partial solution of Question 1.3. Furthermore, they posed the following conjecture without the restriction of isolated fixed-points.

Conjecture 1.4 ([18, Remark H.5, §3, Appendix H]). *Let G be a torus. Then $\beta \in \Omega_*^{U,G}$ is zero if and only if all integral equivariant cohomology Chern numbers of β vanish.*

1.3. Main results. The motivation of this paper is mainly stimulated by the work of Guillemin–Ginzburg–Karshon as mentioned above. First, by using the universal toric genus and the Kronecker pairing of bordism and cobordism which exploit the Quillen’s geometric interpretation of homotopic unitary cobordism theory, we shall give an affirmative answer to Conjecture 1.4, and the result is stated as follows.

Theorem 1.5. *Conjecture 1.4 holds.*

As a further consequence, we obtain a satisfactory solution of Question 1.3.

Corollary 1.6. *Mixed equivariant characteristic numbers separate equivariant Hamiltonian bordism.*

An interesting thing is that our approach above can also be applied to the study of $(\mathbb{Z}_2)^k$ -equivariant unoriented bordism, and in particular, it can still work out the classical result of tom Dieck in [30], which states that the $(\mathbb{Z}_2)^k$ -equivariant unoriented bordism class of a smooth closed $(\mathbb{Z}_2)^k$ -manifold is determined by its $(\mathbb{Z}_2)^k$ -equivariant Stiefel–Whitney numbers. This argument avoids the use of Boardman map.

In addition, we shall also prove the equivalence of integral equivariant cohomology Chern numbers and equivariant K-theoretic Chern numbers for closed unitary G -manifolds by using the developed equivariant Riemann–Roch relation of Atiyah–Hirzebruch type (see Theorem 3.11), so that, together with Theorem 1.5, we may obtain an equivariant version of Theorem 1.1 as follows.

Theorem 1.7. *Let G be a torus. Then the following three statements are equivalent.*

- (1) $\beta \in \Omega_*^{U,G}$ is zero.
- (2) All integral equivariant cohomology Chern numbers of β vanish.
- (3) All equivariant K-theoretic Chern numbers of β vanish.

Remark 1. Clearly, Theorem 1.7 tells us that in a different way we actually obtain the Theorem 1.2 of tom Dieck in the case where G is a torus.

This paper is organized as follows. In Section 2, we shall review the Quillen’s geometric interpretation of homotopic unitary cobordism theory, which is essentially important, so that the Kronecker pairing between bordism and cobordism can be calculated in a geometric way. We also review the universal toric genus defined by Buchstaber–Panov–Ray, which can be expressed in the sense of Quillen’s geometric interpretation. Then we shall give the proof of Theorem 1.5, which is more geometric. In particular, our approach in unitary case can also be carried out very well in the study of $(\mathbb{Z}_2)^k$ -equivariant unoriented bordism, as we shall see in the final part of Section 2. In Section 3, we formulate out an equivariant version of Riemann–Roch relation of Atiyah–Hirzebruch type, and then we use it to prove the equivalence of integral equivariant cohomology Chern numbers and equivariant K-theoretic Chern numbers for closed unitary G -manifolds. Section 4 is an

appendix on (equivariant) Gysin maps, in which we list some known facts needed in this paper.

2. EQUIVARIANT UNITARY BORDISM AND INTEGRAL EQUIVARIANT COHOMOLOGY CHERN NUMBERS

2.1. Geometric interpretation of elements in $MU_*(X)$ and $MU^*(X)$ and Kronecker pairing. Given a topological space X , let $MU_*(X)$ and $MU^*(X)$ be the complex (homotopic) bordism and cobordism of X , which are defined as

$$MU_*(X) = \lim_{l \rightarrow \infty} [S^{2l+*}, X_+ \wedge MU(l)]$$

and

$$MU^*(X) = \lim_{l \rightarrow \infty} [S^{2l-*} \wedge X_+, MU(l)]$$

respectively, where X_+ denotes the union of X and a disjoint point, and $MU(l)$ denotes the Thom space of the universal complex l -dimensional vector bundle over $BU(l)$.

Geometrically, it is known very well that elements of $MU_n(X)$ are regarded as bordism classes of maps $M \rightarrow X$ of stably almost complex closed n -dimensional manifolds M to X . On the geometric interpretation of elements in $MU^*(X)$, Quillen showed in [27, Proposition 1.2] that for a manifold X , $MU^{\pm n}(X)$ is isomorphic to the group formed by cobordism classes of proper complex oriented maps $f : M \rightarrow X$ of dimension $\mp n$, where $\dim X - \dim M = \pm n$. When n is even, a *complex oriented map* $f : M \rightarrow X$ of dimension $\mp n$ is a composition map of manifolds

$$(2.1) \quad M \xrightarrow{i} E \xrightarrow{\pi} X$$

where $\pi : E \rightarrow X$ is a complex vector bundle over X , and $i : M \rightarrow E$ is an embedding endowed with a complex structure on its normal bundle. When n is odd, the complex orientation of f will be defined in a same way as above with E replaced by $E \times \mathbb{R}$ in (2.1).

It is known very well that as generalized cohomology and homology with values in the Thom spectrum MU , $MU^*(X)$ and $MU_*(X)$ can produce a canonical Kronecker pairing

$$\langle \ , \ \rangle : MU^n(X) \otimes MU_m(X) \rightarrow MU_{m-n}$$

where $MU_{m-n} = MU_{m-n}(pt) \cong \Omega_{m-n}^U$.

The Kronecker pairing can be calculated in the geometric way as above. For example, if $\alpha \in MU^{-n}(X)$ is represented by a smooth fiber bundle $E \rightarrow X$ with $\dim E - \dim X = n$ and $\beta \in MU_m(X)$ is represented by a smooth map $f : M \rightarrow X$, then the Kronecker pairing $\langle \alpha, \beta \rangle \in \Omega_{m-n}^U$ is the bordism class of the pull-back $\tilde{f}^*(E) = E \times_X M$ as shown in the following diagram

$$\begin{array}{ccc} E \times_X M & \xrightarrow{\tilde{f}} & E \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

(cf [11, D.3.4, Appendix D]).

2.2. The universal toric genus. Based upon the works of tom Dieck, Krichever and Löffler (see [29, 31, 22, 23]), Buchstaber–Panov–Ray in [10] defined the *universal toric genus* Φ in a geometric manner, which is a ring homomorphism from the geometric unitary T^k -bordism ring to the complex cobordism ring of the classifying space BT^k

$$\Phi : \Omega_*^{U, T^k} \longrightarrow MU^*(BT^k).$$

Note that as a free Ω_*^U -module, $MU^*(BT^k)$ is isomorphic to $\Omega_*^U[[u_1, \dots, u_k]]$ where u_i is the cobordism Chern class $c_1^{MU}(\xi_i)$ of the conjugate Hopf bundle ξ_i over the i -th factor of BT^k for $i = 1, \dots, k$, so $MU^*(BT^k)$ can be replaced by $\Omega_*^U[[u_1, \dots, u_k]]$.

Let $[M]_{T^k} \in \Omega_n^{U, T^k}$ be an element represented by a closed unitary T^k -manifold M . Buchstaber–Panov–Ray showed that $\Phi([M]_{T^k})$ can be defined to be the cobordism class of the complex oriented map $\pi : ET^k \times_{T^k} M \longrightarrow BT^k$. More precisely, choose a T^k -equivariant embedding $i : M \hookrightarrow V$ into a unitary T^k -representation space V . Then the Borelification of i gives a complex oriented map

$$\pi_l : ET^k(l) \times_{T^k} M \hookrightarrow ET^k(l) \times_{T^k} V \longrightarrow BT^k(l)$$

which determines a cobordism class α_l in $MU^{-n}(BT^k(l))$, where $BT^k(l) = (\mathbb{C}P^l)^k$ and $ET^k(l) = (S^{2l+1})^k$. Since $BT^k = \bigcup_l BT^k(l)$ and $ET^k = \bigcup_l ET^k(l)$, these cobordism classes α_l form an inverse system. This produces a class $\alpha = \varprojlim \alpha_l$ in $MU^{-n}(BT^k)$, which is represented geometrically by the complex oriented map

$$\pi : ET^k \times_{T^k} M \hookrightarrow ET^k \times_{T^k} V \longrightarrow BT^k.$$

Then $\Phi([M]_{T^k})$ is defined as this limit α .

The following result is essentially due to Hanke and Löffler (see [19, 23]).

Proposition 2.1. *The ring homomorphism $\Phi : \Omega_*^{U, T^k} \longrightarrow MU^*(BT^k)$ is injective.*

Corollary 2.2. *Let M be a closed unitary T^k -manifold, M is null-bordant in Ω_*^{U, T^k} if and only if the complex oriented map $\pi : ET^k \times_{T^k} M \longrightarrow BT^k$ represents the zero element in $MU^*(BT^k)$.*

Remark 2. As stated in [10], Φ is an equivariant version of the universal Hirzebruch genus. Then for $[M]_{T^k} \in \Omega_n^{U, T^k}$, $\Phi([M]_{T^k})$ can be formally written as

$$\Phi([M]_{T^k}) = \sum_{\omega} g_{\omega}(M) u^{\omega} \text{ in } \Omega_*^U[[u_1, \dots, u_k]]$$

where $\omega = (j_1, \dots, j_k)$ is a partition with $j_i \geq 0$, $u^{\omega} = u_1^{j_1} \cdots u_k^{j_k}$ and the coefficients $g_{\omega}(M) \in \Omega_*^U$. Löffler's work shows in [23, Satz 3.2] that $g_{\omega}(M)$ may be explicitly represented by the manifold $\Gamma^{\omega}(M) := \Gamma_1^{j_1} \cdots \Gamma_k^{j_k}(M)$, where each Γ_i is a composable operation, which is defined as follows: let $S^3 \subset \mathbb{C}^2$ admit a S^1 -action given by $(t, (z_1, z_2)) \mapsto (tz_1, t^{-1}z_2)$, and let the T^k -action on M restrict to the i -th coordinate circle for $i = 1, \dots, k$. Then $\Gamma_i(M)$ is defined to be $M \times_{S^1} S^3$. In [11, §9.2], by employing the null-bordant bounded flag manifolds in Ω_*^U to realize a dual basis in $MU_*(BT^k)$ of the basis $\{u^{\omega}\}$ in $MU^*(BT^k) \cong \Omega_*^U[[u_1, \dots, u_k]]$, the coefficients $g_{\omega}(M)$ can also be represented by unitary manifolds $G_{\omega}(M) = (S^3)^{\omega} \times_{T^{\omega}} M$, where $T^{\omega} = T^{j_1} \times \cdots \times T^{j_k}$ acts coordinatewise on $(S^3)^{\omega} = (S^3)^{j_1} \times \cdots \times (S^3)^{j_k}$ and on M via the representation $(t_{1,1}, \dots, t_{1,j_1}; \dots; t_{k,1}, \dots, t_{k,j_k}) \mapsto (t_{1,j_1}, \dots, t_{k,j_k})$.

2.3. Equivariant Chern classes and equivariant Chern numbers. Let M be a closed unitary T^k -manifold. Then applying the Borel construction to the tangent bundle TM of M gives a T^k -vector bundle $ET^k \times_{T^k} TM$ over $ET^k \times_{T^k} M$.

Definition 2.3. The *total equivariant cohomology Chern class* of M is defined to be the total Chern class of the vector bundle $ET^k \times_{T^k} TM$ over $ET^k \times_{T^k} M$, i.e.

$$\begin{aligned} c^{T^k}(M) &:= c(ET^k \times_{T^k} TM) \\ &= 1 + c_1(ET^k \times_{T^k} TM) + c_2(ET^k \times_{T^k} TM) + \cdots \\ &= 1 + c_1^{T^k}(M) + c_2^{T^k}(M) + \cdots \in H^*(ET^k \times_{T^k} M) = H_{T^k}^*(M). \end{aligned}$$

Let $p : M \rightarrow pt$ be the constant map, and let $p_!^{T^k} : H^*(ET^k \times_{T^k} M) \rightarrow H^*(BT^k)$ be the equivariant Gysin map induced by p . Therefore, the *integral equivariant cohomology Chern numbers* of M are defined to be:

$$c_\omega^{T^k}[M]_{T^k} := p_!^{T^k}(c_\omega^{T^k}(M)),$$

each of which is a homogeneous polynomial of degree $2|\omega| - \dim M$ in the polynomial ring $H^*(BT^k) = \mathbb{Z}[x_1, \dots, x_k]$ with $\deg x_i = 2$, where $\omega = (i_1, i_2, \dots, i_s)$ is a partition of $|\omega| = i_1 + i_2 + \cdots + i_s$, and $c_\omega^{T^k}(M)$ means the product $c_{i_1}^{T^k}(M)c_{i_2}^{T^k}(M) \cdots c_{i_s}^{T^k}(M)$.

Note that the map $\pi : ET^k \times_{T^k} M \rightarrow BT^k$ is the Borelification of $p : M \rightarrow pt$, so $p_!^{T^k}$ is often replaced by the Gysin map $\pi_!$ in this paper.

Remark 3. According to [16, 17], the integral equivariant cohomology Chern numbers $c_\omega^{T^k}[M]_{T^k}$ are also called the generalised (unitary) Miller–Morita–Mumford classes of the fibration $\pi : ET^k \times_{T^k} M \rightarrow BT^k$.

2.4. Proof of Theorem 1.5. Let M be a closed unitary T^k -manifold. Assume that all integral equivariant cohomology Chern numbers of M vanish. In order to prove that M is null-bordant in Ω_*^{U, T^k} , it suffices to show that the image $\Phi([M]_{T^k})$ of the universal toric genus

$$\Phi : \Omega_*^{U, T^k} \rightarrow MU^*(BT^k)$$

vanishes since the genus Φ is injective.

Geometrically, the image $\Phi([M]_{T^k})$ is represented by the map $\pi : ET^k \times_{T^k} M \rightarrow BT^k$. Consider the Kronecker pairing

$$MU^*(BT^k) \otimes MU_*(BT^k) \rightarrow MU_*.$$

Let $[f : X \rightarrow BT^k]$ be an element in $MU_*(BT^k)$, where X is a closed unitary manifold. Then the Kronecker pairing between $\Phi([M]_{T^k})$ and $[f : X \rightarrow BT^k]$ is represented by the pull-back of the fibration $\pi : ET^k \times_{T^k} M \rightarrow BT^k$ via f

$$\begin{array}{ccc} \tilde{f}^*(ET^k \times_{T^k} M) & \xrightarrow{\tilde{f}} & ET^k \times_{T^k} M \\ \pi' \downarrow & & \downarrow \pi \\ X & \xrightarrow{f} & BT^k \end{array}$$

Namely, the Kronecker pairing between $\Phi([M]_{T^k})$ and $[f : X \rightarrow BT^k]$ is

$$\langle \Phi([M]_{T^k}), [f : X \rightarrow BT^k] \rangle = [\tilde{f}^*(ET^k \times_{T^k} M)] \in MU_*.$$

Since $BT^k = \bigcup_n BT^k(n)$ with $BT^k(n) = (\mathbb{C}P^n)^k$, the compactness of X makes sure that there exists some positive integer m such that $f : X \rightarrow BT^k$ factors through $i_m \circ f_m$

$$f : X \xrightarrow{f_m} BT^k(m) \xrightarrow{i_m} BT^k$$

where i_m is a natural inclusion. Thus, the fibration $\pi' : \tilde{f}^*(ET^k \times_{T^k} M) \rightarrow X$ factors through

$$\begin{array}{ccccccc} \tilde{f}^*(ET^k \times_{T^k} M) & \xlongequal{\quad} & \tilde{f}_m^*(ET^k(m) \times_{T^k} M) & \xrightarrow{\tilde{f}_m} & ET^k(m) \times_{T^k} M & \longrightarrow & ET^k \times_{T^k} M \\ \pi' \downarrow & & \pi' \downarrow & & \pi \downarrow & & \pi \downarrow \\ X & \xlongequal{\quad} & X & \xrightarrow{f_m} & BT^k(m) & \xrightarrow{i_m} & BT^k. \end{array}$$

This implies that the pull-back $\tilde{f}^*(ET^k \times_{T^k} M) = \tilde{f}_m^*(ET^k(m) \times_{T^k} M)$ is a closed unitary manifold and has dimension $\dim M + \dim X$. In particular, we see easily that if $X = \mathbb{C}P^{n_1} \times \cdots \times \mathbb{C}P^{n_k}$ with $n_i \geq 0$, then

$$\tilde{f}^*(ET^k \times_{T^k} M) = \left(\prod_{i=1}^k S^{2n_i+1} \right) \times_{T^k} M.$$

With the above understanding, we proceed our argument as follows.

Lemma 2.4. *Suppose that the dimension of M is positive and all integral equivariant cohomology Chern numbers of M vanish, then the class $[\tilde{f}^*(ET^k \times_{T^k} M)] = 0 \in \Omega_*^U$.*

Proof. Since $\tilde{f}^*(ET^k \times_{T^k} M)$ is the pull-back of the fibration $\pi : ET^k \times_{T^k} M \rightarrow BT^k$, the tangent bundle of $\tilde{f}^*(ET^k \times_{T^k} M)$ is isomorphic to $\pi'^*TX \oplus \tilde{f}^*(ET^k \times_{T^k} TM)$. So the total Chern class of $\tilde{f}^*(ET^k \times_{T^k} M)$ is equal to

$$c(\tilde{f}^*(ET^k \times_{T^k} M)) = \pi'^*(c(X)) \cdot \tilde{f}^*(c(ET^k \times_{T^k} TM)),$$

and one then has

$$c_i(\tilde{f}^*(ET^k \times_{T^k} M)) = \tilde{f}^*(c_i^{T^k}(M)) + \pi'^*(c_i(X)) + \sum_{\substack{l+j=i \\ l>0, j>0}} \pi'^*(c_l(X)) \cdot \tilde{f}^*(c_j^{T^k}(M)).$$

Taking a partition $\omega = (i_1, \dots, i_s)$ with $|\omega| = \dim \tilde{f}^*(ET^k \times_{T^k} M)$, the product

$$c_\omega(\tilde{f}^*(ET^k \times_{T^k} M)) = c_{i_1}(\tilde{f}^*(ET^k \times_{T^k} M)) \cdots c_{i_s}(\tilde{f}^*(ET^k \times_{T^k} M))$$

can further be written as

$$(2.2) \quad c_\omega(\tilde{f}^*(ET^k \times_{T^k} M)) = \tilde{f}^*(c_\omega^{T^k}(M)) + \pi'^*(c_\omega(X)) + \sum_j \pi'^*(\beta_j) \cdot \gamma_j$$

where β_j is the product of some Chern classes of X with $\deg \beta_j \neq 0$ and γ_j is the product of the pull-back via \tilde{f}^* of some equivariant Chern classes of M with $\deg \gamma_j \neq 0$.

Let $p : X \rightarrow pt$ be the constant map. Then we have by Proposition 4.2(3) that the integral cohomology Chern number of $\tilde{f}^*(ET^k \times_{T^k} M)$ determined by the class $c_\omega(\tilde{f}^*(ET^k \times_{T^k} M))$ is equal to

$$c_\omega[\tilde{f}^*(ET^k \times_{T^k} M)] = (p\pi')!(c_\omega(\tilde{f}^*(ET^k \times_{T^k} M))) = p!\pi'_!(c_\omega(\tilde{f}^*(ET^k \times_{T^k} M))).$$

Now, for the class $\tilde{f}^*(c_\omega^{T^k}(M))$ in (2.2), one has by Proposition 4.5 that

$$\pi'_!(\tilde{f}^*(c_\omega^{T^k}(M))) = f^*(\pi_!(c_\omega^{T^k}(M))) = 0$$

since we have assumed that all integral equivariant Chern numbers of M are zero. For the class $\pi'^*(c_\omega(X))$ in (2.2), since the dimension of $f^*(ET^k \times_{T^k} M)$ is strictly bigger than the dimension of X , one has by Proposition 4.2(1)-(2) that $\pi'_!(1) = 0$ and

$$\pi'_!(\pi'^*(c_\omega(X))) = \pi'_!(\pi'^*(c_\omega(X)) \cdot 1) = c_\omega(X) \cdot \pi'_!(1) = 0.$$

For the class $\pi'^*(\beta_j) \cdot \gamma_j$ in (2.2), we may write $\gamma_j = \tilde{f}^*(c_{\omega'}^{T^k}(M))$ for some partition ω' with $|\omega'| \neq 0$. Then one has by Propositions 4.2 and 4.5 that

$$\pi'_!(\pi'^*(\beta_j) \cdot \gamma_j) = \beta_j \cdot \pi'_!(\gamma_j) = \beta_j \cdot \pi'_!(\tilde{f}^*(c_{\omega'}^{T^k}(M))) = \beta_j \cdot f^*(\pi_!(c_{\omega'}^{T^k}(M))) = 0$$

since $\pi_!(c_{\omega'}^{T^k}(M)) = 0$ by hypothesis. Thus, we have that

$$\begin{aligned} \pi'_!(c_\omega(\tilde{f}^*(ET^k \times_{T^k} M))) &= \pi'_!(\tilde{f}^*(c_\omega^{T^k}(M)) + \pi'^*(c_\omega(X)) + \sum_j \pi'^*(\beta_j) \cdot \gamma_j) \\ &= \pi'_!(\tilde{f}^*(c_\omega^{T^k}(M))) + \pi'_!(\pi'^*(c_\omega(X))) + \pi'_!(\sum_j \pi'^*(\beta_j) \cdot \gamma_j) \\ &= 0 \end{aligned}$$

so $(p\pi')_!(c_\omega(\tilde{f}^*(ET^k \times_{T^k} M))) = 0$. This shows that all integral cohomology Chern numbers of $\tilde{f}^*(ET^k \times_{T^k} M)$ vanish. Therefore, the bordism class $[\tilde{f}^*(ET^k \times_{T^k} M)] = 0$ in Ω_*^{U,T^k} by the classical results of Milnor and Novikov (cf [25, 26]). \square

Proposition 2.5. *Let M be a closed unitary T^k -manifold. Assume that all integral equivariant cohomology Chern numbers of M vanish, then $[M]_{T^k} = 0 \in \Omega_*^{U,T^k}$.*

Proof. First, when $\dim M = 0$, we know that the T^k -action on M is trivial and proposition holds in this case.

Now suppose that $\dim M > 0$, by Lemma 2.4, one sees that for any element $\alpha \in MU_*(BT^k)$, the Kronecker pairing

$$\langle \Phi([M]_{T^k}), \alpha \rangle = 0 \in MU_*.$$

According to [1, Page 48], $MU^*(BT^k) = \text{Hom}_{MU_*}(MU_*(BT^k), MU_*)$. It follows that $\Phi([M]_{T^k}) = 0 \in MU^*(BT^k)$. Hence, by Corollary 2.2, one has $[M]_{T^k} = 0 \in \Omega_*^{U,T^k}$. \square

On the other hand, we can show

Proposition 2.6. *If $[M]_{T^k} = 0 \in \Omega_*^{U,T^k}$, then all integral equivariant cohomology Chern numbers of M vanish.*

Proof. If $\dim M = 0$, this is trivial. Assume $\dim M > 0$. Since $[M]_{T^k} = 0 \in \Omega_*^{U,T^k}$, one has $\Phi([M]_{T^k}) = [\pi : ET^k \times_{T^k} M \rightarrow BT^k] = 0 \in MU^*(BT^k)$. Given a Chern class $c_\omega^{T^k}(M) \in H^{2\mathcal{N}}(ET^k \times_{T^k} M)$ where $\omega = (i_1, i_2, \dots, i_s)$ with $|\omega| = \mathcal{N} \in \mathbb{Z}_{\geq 0}$. If $\dim M = 2m + 1$, then the corresponding integral equivariant cohomology Chern number

$$\pi_!(c_\omega^{T^k}(M)) \in H^{2\mathcal{N}-2m-1}(BT^k) = 0.$$

Hence, we assume that $\dim M = 2m > 0$ and we have

$$\pi_!(c_\omega^{T^k}(M)) \in H^{2\mathcal{N}-2m}(BT^k).$$

To complete the proof, it suffices to show that for arbitrary $\mathcal{N} \geq 0$, the integral equivariant cohomology Chern number

$$\pi_!(c_\omega^{T^k}(M)) = 0.$$

Clearly, if $\mathcal{N} < m$, then $\pi_!(c_\omega^{T^k}(M)) = 0$. Next we shall perform an induction on $\mathcal{N} - m \geq 0$.

When $\mathcal{N} - m = 0$, we have that $\pi_!(c_\omega^{T^k}(M)) = l \in H^0(BT^k) \cong \mathbb{Z}$. In this case, $\pi_!(c_\omega^{T^k}(M))$ is actually an ordinary nonequivariant Chern number $c_\omega[M]$. Let $X = \{pt\}$ be a point. Consider the map $f : X \rightarrow BT^k$, one sees that $f(X)$ is a point of BT^k , denoted by b . Then $\tilde{f}^*(ET^k \times_{T^k} M) = X \times \pi^{-1}(b)$ is exactly homeomorphic to M . Since $\Phi([M]_{T^k}) = 0$, one has that the Kronecker pairing

$$[M] = [\tilde{f}^*(ET^k \times_{T^k} M)] = \langle \Phi([M]_{T^k}), [f : X \rightarrow BT^k] \rangle = 0 \text{ in } MU_{2m}$$

so all Chern numbers of M are zero. Thus, $\pi_!(c_\omega^{T^k}(M)) = l = 0$ as desired.

Now we assume inductively that if $\mathcal{N} - m \leq \ell$ (i.e., $\deg c_\omega^{T^k}(M) = 2\mathcal{N} \leq 2m + 2\ell$), then $\pi_!(c_\omega^{T^k}(M)) = 0$. When $\mathcal{N} - m = \ell + 1$, since $H^*(BT^k) \cong \mathbb{Z}[x_1, x_2, \dots, x_k]$ with $\deg x_i = 2$, one can write

$$\pi_!(c_\omega^{T^k}(M)) = \sum_J n_J x^J,$$

where $J = (j_1, j_2, \dots, j_k)$ ranges over all partitions with $|J| = j_1 + j_2 + \dots + j_k = \mathcal{N} - m = \ell + 1$ and $j_l \geq 0$ for $l = 1, 2, \dots, k$, x^J means $x_1^{j_1} x_2^{j_2} \dots x_k^{j_k}$, and $n_J \in \mathbb{Z}$. Our next task is to show that for all J , $n_J = 0$.

Take a partition $I = (j_1, j_2, \dots, j_k)$ with $|I| = \mathcal{N} - m$ and set $X = \mathbb{C}P^{j_1} \times \mathbb{C}P^{j_2} \times \dots \times \mathbb{C}P^{j_k}$. It is well-known that the cohomology ring

$$H^*(X) = \mathbb{Z}[t_1, t_2, \dots, t_k] / (t_i^{j_i+1} = 0, i = 1, \dots, k)$$

with $\deg t_i = 2, i = 1, \dots, k$. Consider the natural embedding

$$f_I : X \hookrightarrow BT^k.$$

It is easy to see that $f_I^* : H^2(BT^k) \rightarrow H^2(X)$ is isomorphic, so we may assume without loss of generality that $f_I^*(x_i) = t_i, i = 1, 2, \dots, k$.

On the other hand, we see that the bordism class of the pull-back $\tilde{f}_I^*(ET^k \times_{T^k} M)$ vanishes in MU_* , i.e.

$$\langle \Phi([M]_{T^k}), [f_I : X \hookrightarrow BT^k] \rangle = [\tilde{f}_I^*(ET^k \times_{T^k} M)] = 0 \in MU_*$$

since we have known that $\Phi([M]_{T^k}) = 0$. This implies that all integral cohomology Chern numbers of $\tilde{f}_I^*(ET^k \times_{T^k} M)$ vanish. Then we have that the Chern class

$$c_\omega(\tilde{f}_I^*(ET^k \times_{T^k} M)) = 0$$

in $H^*(\tilde{f}_I^*(ET^k \times_{T^k} M))$ since $\deg c_\omega(\tilde{f}_I^*(ET^k \times_{T^k} M)) = 2\mathcal{N} = \dim \tilde{f}_I^*(ET^k \times_{T^k} M)$. Note that $\tilde{f}_I^*(ET^k \times_{T^k} M) = (\prod_{l=1}^k S^{2j_l+1}) \times_{T^k} M$. By the proof of Lemma 2.4, we can write

$$0 = c_\omega(\tilde{f}_I^*(ET^k \times_{T^k} M)) = \tilde{f}_I^*(c_\omega^{T^k}(M)) + \pi'^*(c_\omega(X)) + \sum_j \pi'^*(\beta_j) \cdot \gamma_j,$$

where π' denotes the map $\tilde{f}_I^*(ET^k \times_{T^k} M) \rightarrow X$ and β_j, γ_j have the same meanings as stated in the proof of Lemma 2.4. So one has that

$$\begin{aligned} 0 &= \pi'_!(c_\omega(\tilde{f}_I^*(ET^k \times_{T^k} M))) \\ &= \pi'_!(\tilde{f}_I^*(c_\omega^{T^k}(M)) + \pi'^*(c_\omega(X)) + \sum_j \pi'^*(\beta_j) \cdot \gamma_j) \\ &= \pi'_!(\tilde{f}_I^*(c_\omega^{T^k}(M))) + \pi'_!(\pi'^*(c_\omega(X))) + \pi'_!(\sum_j \pi'^*(\beta_j) \cdot \gamma_j). \end{aligned}$$

In the above equation, we see that the term

$$\pi'_!(\pi'^*(c_\omega(X))) = c_\omega(X) \cdot \pi'_!(1) = 0,$$

and we have by induction hypothesis that the term

$$\pi'_!(\sum_j \pi'^*(\beta_j) \cdot \gamma_j) = \sum_j \beta_j \cdot \pi'_!(\gamma_j) = 0$$

since $\deg \gamma_j \leq 2\mathcal{N} - 2 = 2m + 2\ell$. This gives that $\pi'_!(\tilde{f}_I^*(c_\omega^{T^k}(M))) = 0$. Therefore, we have by Proposition 4.5 that

$$0 = \tilde{f}_I^*(\pi_!(c_\omega^{T^k}(M))) = \tilde{f}_I^*(\sum_J n_J x^J) = \sum_J n_J \tilde{f}_I^*(x^J) = \sum_J n_J t^J = n_I t^I$$

in $H^{2\mathcal{N}-2m}(X)$. Because $t^I = t_1^{j_1} t_2^{j_2} \cdots t_k^{j_k} \neq 0$ in $H^{2\mathcal{N}-2m}(X)$, one has $n_I = 0$. This means that all the integral equivariant cohomology Chern numbers $\pi_!(c_\omega^{T^k}(M))$ of M vanish. \square

Together with the above arguments, one has

Theorem 2.7. *Let M be a closed unitary T^k -manifold. Then the following statements are equivalent.*

- (1) M is null-bordant in Ω_*^{U, T^k} .
- (2) All integral equivariant cohomology Chern numbers of M vanish.
- (3) For any partition $I = (j_1, \dots, j_k)$ with each $j_i \geq 0$, $(\prod_{i=1}^k S^{2j_i+1}) \times_{T^k} M$ is null-bordant in Ω_*^U .
- (4) The fibration $\pi : ET^k \times_{T^k} M \rightarrow BT^k$ is null-cobordant in $MU^*(BT^k)$.

2.5. Equivariant unoriented bordism. Our approach above can also be carried out in the case of equivariant unoriented (co)bordism. Let $\mathfrak{N}_*^{(\mathbb{Z}_2)^k}$ be the ring formed by the equivariant bordism classes of all unoriented closed smooth $(\mathbb{Z}_2)^k$ -manifolds, and let $MO^*(X)$ be the unoriented (homotopic) cobordism ring of a topological space X , i.e.,

$$MO^*(X) = \lim_{l \rightarrow \infty} [S^{l-*} \wedge X_+, MO(l)].$$

Then Quillen's geometric approach on $MU^*(X)$ can be carried out to the case of $MO^*(X)$, so that the Kronecker pairing

$$\langle \ , \ \rangle : MO^*(X) \otimes MO_*(X) \longrightarrow MO_* \cong \mathfrak{N}_*$$

can be calculated in a similar geometric way to the unitary case as in Subsection 2.1 (cf [11, D.2.8, D.3.4, Appendix D]), where \mathfrak{N}_* is the nonequivariant Thom unoriented bordism ring. In [30], tom Dieck showed that there is also a monomorphism

$$\Phi_{\mathbb{R}} : \mathfrak{N}_*^{(\mathbb{Z}_2)^k} \longrightarrow MO^*(B(\mathbb{Z}_2)^k).$$

It is well-known that $MO^*(B(\mathbb{Z}_2)^k) = \varprojlim MO^*(B(\mathbb{Z}_2)^k(n))$ where $B(\mathbb{Z}_2)^k(n) = (\mathbb{R}P^n)^k$ and $B(\mathbb{Z}_2)^k = \cup_n B(\mathbb{Z}_2)^k(n)$. Then, applying the finite approximation method may yield that for a class $[M]_{(\mathbb{Z}_2)^k} \in \mathfrak{N}_*^{(\mathbb{Z}_2)^k}$, the image $\Phi_{\mathbb{R}}([M]_{(\mathbb{Z}_2)^k})$ is geometrically represented by the fibration $\pi : E(\mathbb{Z}_2)^k \times_{(\mathbb{Z}_2)^k} M \longrightarrow B(\mathbb{Z}_2)^k$.

Theorem 2.8. *Let M be a closed smooth $(\mathbb{Z}_2)^k$ -manifold. Then the following statements are equivalent.*

- (1) *M is null-bordant in $\mathfrak{N}_*^{(\mathbb{Z}_2)^k}$.*
- (2) *All equivariant Stiefel–Whitney numbers of M vanish.*
- (3) *For any partition $I = (j_1, \dots, j_k)$ with each $j_i \geq 0$, $(\prod_{i=1}^k S^{j_i+1}) \times_{(\mathbb{Z}_2)^k} M$ is null-bordant in \mathfrak{N}_* .*
- (4) *The fibration $\pi : E(\mathbb{Z}_2)^k \times_{(\mathbb{Z}_2)^k} M \longrightarrow B(\mathbb{Z}_2)^k$ is null-cobordant in $MO^*(B(\mathbb{Z}_2)^k)$.*

Proof. The proof follows closely that of Theorem 2.7 in a very similar way. We would like to leave it as an exercise to the reader. \square

Remark 4. Theorem 2.8 tells us that the $(\mathbb{Z}_2)^k$ -equivariant unoriented bordism is determined by $(\mathbb{Z}_2)^k$ -equivariant Stiefel–Whitney numbers. This result was essentially due to tom Dieck with the argument of involving the Boardman map (see [30]). Here we give another proof without using the Boardman map.

It is known well that as a free \mathfrak{N}_* -module, $MO^*(B(\mathbb{Z}_2)^k)$ is isomorphic to $\mathfrak{N}_*[[v_1, \dots, v_k]]$, where v_i denotes the first cobordism Stiefel–Whitney class of the canonical line bundle over the i -th factor of $B(\mathbb{Z}_2)^k$ for $i = 1, \dots, k$. Thus, in the sense of Buchstaber–Panov–Ray [10], we can define the $(\mathbb{Z}_2)^k$ -equivariant universal genus as follows:

$$\Phi_{\mathbb{R}} : \mathfrak{N}_*^{(\mathbb{Z}_2)^k} \longrightarrow \mathfrak{N}_*[[v_1, \dots, v_k]]$$

by $\Phi_{\mathbb{R}}([M]_{(\mathbb{Z}_2)^k}) = \sum_{\omega=(j_1, \dots, j_k)} g_{\omega}^{\mathbb{R}}(M) v^{\omega}$, where $v^{\omega} = v_1^{j_1} \cdots v_k^{j_k}$ for $\omega = (j_1, \dots, j_k)$ with $j_i \geq 0$, and $g_{\omega}^{\mathbb{R}}(M) \in \mathfrak{N}_*$ with $\dim g_{\omega}^{\mathbb{R}}(M) = |\omega| + \dim M$.

The determination of coefficients $g_{\omega}^{\mathbb{R}}(M)$ depends upon the choices of the dual bases in $MO_*(B(\mathbb{Z}_2)^k)$ of the basis $\{v^{\omega}\}$ in $MO^*(B(\mathbb{Z}_2)^k)$. Buchstaber–Panov–Ray's approach in the unitary case provides us much more insights on the determination of coefficients $g_{\omega}^{\mathbb{R}}(M)$. In our case, since it was shown in [24, Corollary 6.4] that any real Bott manifold is null-bordant in \mathfrak{N}_* , we may employ the real Bott manifolds to realize a dual basis in $MO_*(B(\mathbb{Z}_2)^k)$ of the basis $\{v^{\omega}\}$ in $MO^*(B(\mathbb{Z}_2)^k)$, so that the coefficients $g_{\omega}^{\mathbb{R}}(M)$ can be represented by manifolds $G_{\omega}^{\mathbb{R}}(M) = (S^1)^{\omega} \times_{(\mathbb{Z}_2)^{\omega}} M$, where $(\mathbb{Z}_2)^{\omega} = (\mathbb{Z}_2)^{j_1} \times \cdots \times (\mathbb{Z}_2)^{j_k}$ acts coordinatewise on $(S^1)^{\omega} = (S^1)^{j_1} \times \cdots \times (S^1)^{j_k}$ and on M via the representation

$(g_{1,1}, \dots, g_{1,j_1}; \dots; g_{k,1}, \dots, g_{k,j_k}) \mapsto (g_{1,j_1}, \dots, g_{k,j_k})$. The argument is close to that of the unitary case in [11, §9.2]. We would like to leave it to the reader.

3. THE EQUIVALENCE OF INTEGRAL EQUIVARIANT COHOMOLOGY CHERN NUMBERS AND EQUIVARIANT K-THEORETIC CHERN NUMBERS

In this section, we first deal with the case of nonequivariant unitary bordism, and give a simple proof of the equivalence of integral cohomology Chern numbers and K-theoretic Chern numbers for determining the unitary bordism class of a closed unitary manifold. Next we formulate out an equivariant version of Riemann–Roch relation of Atiyah–Hirzebruch type, and we then use it to show the equivalence of integral equivariant cohomology Chern numbers and equivariant K-theoretic Chern numbers for closed unitary T^k -manifolds.

3.1. K -theoretic Chern class and Chern number. First let us review the definitions of K -theoretic Chern class and Chern number and some standard facts (cf [3]). Let ξ be a complex vector bundle over a space X and $\lambda_t(\xi) \in K^*(X)[[t]]$ be the power series:

$$\sum_{i=0}^{\infty} \lambda^i(\xi) t^i$$

where $\lambda^i(\xi)$ denotes the i -th exterior power of the vector bundle ξ . Then λ_t can be extended to a homomorphism:

$$\begin{aligned} \lambda_t : K^*(X) &\longrightarrow K^*(X)[[t]] \\ x &\mapsto \sum_{i=0}^{\infty} \lambda^i(x) t^i. \end{aligned}$$

Furthermore, one can define the operations $\gamma^i(x)$ by putting

$$\gamma_t(x) := \lambda_{t/1-t}(x) = \sum_i \gamma^i(x) t^i \in K(X)[[t]], \quad x \in K(X).$$

Proposition 3.1 ([3]). *Let ξ and η be two complex bundles over a finite CW complex X . One has:*

- (a) *If ξ is a line bundle, then $\gamma^1(\xi) = \xi - 1$ and $\gamma^k(\xi) = 0$ for $k > 1$.*
- (b) *$\gamma^k(\xi \oplus \eta) = \sum_{p+q=k} \gamma^p(\xi) \cdot \gamma^q(\eta)$.*
- (c) *Let $f : Y \rightarrow X$ be a continuous map. Then $f^! \gamma^k(\xi) = \gamma^k(f^!(\xi))$.*

Now we can use these operations γ^i to define K -theoretic Chern classes. Let M be a closed unitary manifold with stable almost complex tangent bundle $TM \in K(M)$. Then the *total K -theoretic Chern class* of M is defined as

$$c^K(M) := \sum_{i=0}^{\infty} \gamma^i(TM)$$

with the i -th K -theoretic Chern class $c_i^K(M) = \gamma^i(TM)$.

Remark 5. In [15], the i -th K -theoretic Chern class of M is also defined by $(-1)^i \gamma^i(TM)$. In this paper, it is more convenient to use the definition of $\gamma^i(TM)$ as above.

Using the Gysin map, we can define the K -theoretic Chern numbers of M . Namely, let $p : M \rightarrow pt$ be the constant map and $p_!^K : K^*(M) \rightarrow K^*(pt)$ be the Gysin map induced by p in complex K -theory. Then the K -theoretic Chern numbers of M are defined to be:

$$c_\omega^K[M] := p_!^K(c_\omega^K(M)),$$

where each $\omega = (i_1, i_2, \dots, i_s)$ is a partition of $|\omega| = i_1 + i_2 + \dots + i_s$, and $c_\omega^K(M)$ means $c_{i_1}^K(M)c_{i_2}^K(M) \cdots c_{i_s}^K(M)$.

3.2. Riemann–Roch relations. In this part, we shall review the Riemann–Roch relations (cf [5, 21]).

Definition 3.2. For any topological space X , we denote $H^{**}(X; R)$ by the direct product of $H^i(X; R)$ with coefficient ring R . More precisely,

$$H^{**}(X; R) := \{(x_0, x_1, \dots, x_n, \dots) \mid x_i \in H^i(X; R)\}.$$

For any two elements $a = \{x_i\}$, $b = \{y_i\}$ in $H^{**}(X; R)$, the product $a \cdot b = \{z_i\}$ is defined to be:

$$z_n = \sum_{i=0}^n x_i y_{n-i}.$$

Remark 6. When we discuss the case of the Borel construction $EG \times_G X$ (which is not compact) of X , it should be better to use direct product $H_G^{**}(X; \mathbb{Q})$ instead of cohomology ring $H_G^*(X; \mathbb{Q}) = \bigoplus H_G^i(X; \mathbb{Q})$.

We know that Chern character ch is a natural transformation from complex K -theory to ordinary cohomology theory with rational coefficients:

$$ch : K^*(X) \rightarrow H^{**}(X; \mathbb{Q}).$$

Let X be a compact space and ξ be a complex vector bundle over X . Then one has Thom isomorphisms in complex K -theory and ordinary cohomology theory:

$$\begin{aligned} \psi_! : K^*(X) &\rightarrow K^*(D(\xi), S(\xi)), \\ \psi_* : H^*(X) &\rightarrow H^*(D(\xi), S(\xi)). \end{aligned}$$

With respect to these Thom isomorphisms $\psi_!$ and ψ_* , one has the following diagram:

$$\begin{array}{ccc} K^*(X) & \xrightarrow{ch} & H^{**}(X; \mathbb{Q}) \\ \psi_! \downarrow & & \psi_* \downarrow \\ K^*(D(\xi), S(\xi)) & \xrightarrow{ch} & H^{**}(D(\xi), S(\xi); \mathbb{Q}). \end{array}$$

This diagram is **NOT** commutative. However, we have the following nonequivariant Riemann–Roch relation.

Proposition 3.3. *Let X be a compact space and ξ be a complex vector bundle over X . Then for each $\alpha \in K^*(X)$,*

$$ch(\psi_!(\alpha)) = \psi_*(ch(\alpha) \cdot Td^{-1}(\xi)),$$

where $Td(\xi) \in H^{**}(X; \mathbb{Q})$ is the total Todd class of ξ .

Proof. We refer to [21, page 182, 24.5] or [6, Proposition 3.5]. □

With respect to the Gysin maps, there is the Riemann–Roch relation of Atiyah–Hirzebruch type (cf [5] or [21, Theorem 26.5.2]). In the case of closed unitary manifolds, this kind of Riemann–Roch relation can be stated as follows:

Proposition 3.4. *Let M and N be two closed unitary manifolds and $f : M \longrightarrow N$ be a map. Then for each $\alpha \in K^*(M)$,*

$$ch(f_!^K(\alpha) \cdot Td(TN)) = f_!(ch(\alpha) \cdot Td(TM))$$

where $f_!^K : K^*(M) \longrightarrow K^{*-(m-n)}(N)$ and $f_! : H^*(M; \mathbb{Q}) \longrightarrow H^{*-(m-n)}(N; \mathbb{Q})$ are the Gysin maps induced by f in complex K -theory and ordinary cohomology theory, respectively, and $m = \dim M$ and $n = \dim N$.

Remark 7. In a special case of Proposition 3.4, let $p : M \longrightarrow pt$ be the constant map. Then for each $\alpha \in K^*(M)$,

$$ch(p_!^K(\alpha)) = p_!(ch(\alpha) \cdot Td(TM)).$$

3.3. K -theoretic Chern numbers and ordinary cohomology Chern numbers. In this part, we will use the Riemann–Roch relation of Atiyah–Hirzebruch type to prove:

Proposition 3.5. *Let M be a closed unitary manifold with stable almost complex tangent bundle TM . All integral cohomology Chern numbers vanish if and only if all K -theoretic Chern numbers vanish.*

Proof. First, assume all integral cohomology Chern numbers of M vanish. Then for any K -theoretic Chern number $p_!^K(c_\omega^K(M))$ with partition $w = (i_1, i_2, \dots, i_s)$, one has:

$$ch(p_!^K(c_\omega^K(M))) = p_!(ch(c_\omega^K(M)) \cdot Td(TM)),$$

where $p_! : H^{**}(M; \mathbb{Q}) \longrightarrow H^{**}(pt; \mathbb{Q})$ is the Gysin map induced by p in ordinary cohomology. Since $p_!(ch(c_\omega^K(M)) \cdot Td(TM))$ is represented by a combinations of ordinary cohomology Chern numbers, it follows that

$$p_!(ch(c_\omega^K(M)) \cdot Td(TM)) = 0 = ch(p_!^K(c_\omega^K(M))).$$

Moreover, since the Chern character $ch : K^*(pt) \longrightarrow H^{**}(pt; \mathbb{Q})$ is injective, one has $p_!^K(c_\omega^K(M)) = 0$.

Conversely, assume all K -theoretic Chern numbers of M vanish. Then for any cohomology Chern number $p_!(c_\omega(M))$, consider the corresponding K -theoretic Chern class $c_\omega^K(M)$. One has

$$ch(c_\omega^K(M)) = c_\omega(M) + \text{terms of higher degree} \in H^{**}(M; \mathbb{Q}).$$

By Riemann–Roch relation, one has:

$$\begin{aligned} 0 &= ch(p_!^K(c_\omega^K(M))) \\ &= p_!(ch(c_\omega^K(M)) \cdot Td(TM)) \\ &= p_!(c_\omega(M) + \text{terms of higher degree}) \\ &= p_!(c_\omega(M)) + \text{terms of higher degree} \end{aligned}$$

which induces that $p_!(c_\omega(M)) = 0$ in $H^{**}(pt; \mathbb{Q})$. This means that $p_!(c_\omega(M)) = 0$ in $H^{**}(pt) \cong \mathbb{Z}$ since Ω_*^U is torsion free. The proof is completed. \square

3.4. Equivariant Chern classes and Chern numbers. In this section, we recall the definitions of three kinds of equivariant Chern classes. Let M be a unitary G -manifold and E be a unitary G -vector bundle over M . Then applying the Borel construction to $E \rightarrow M$ gives a unitary G -vector bundle $EG \times_G E$ over $EG \times_G M$. First, recall that the total equivariant cohomology Chern class of E is defined by the total Chern class of $EG \times_G E$ over $EG \times_G M$:

$$c^G(E) := c(EG \times_G E) \in H^{**}(EG \times_G M).$$

Similarly, in the equivariant K -theory $K(EG \times_G -)$, the total equivariant Chern class of E over M is defined to be:

$$c^{G,K}(E) := c^K(EG \times_G E) \in K^*(EG \times_G M).$$

In equivariant K -theory $K_G(-)$, the total equivariant Chern class of E is defined to be:

$$c^{G,K_G}(E) := \sum_{i=0} \gamma^i(E) \in K_G^*(M)$$

where the operations $\gamma^i(E)$ can be defined in a nonequivariant manner as before (cf [20, 31]).

In particular, let M be a unitary G -manifold with stable almost complex G -vector bundle TM . The corresponding total equivariant Chern class of M is defined by the corresponding total equivariant Chern class of TM as follows:

$$\begin{aligned} c^G(M) &:= c(EG \times_G TM) \in H^{**}(EG \times_G M); \\ c^{G,K}(M) &:= c^K(EG \times_G TM) \in K^*(EG \times_G M); \\ c^{G,K_G}(M) &:= \sum_{i=0} \gamma^i(TM) \in K_G^*(M). \end{aligned}$$

We know the equivariant Gysin maps are well-defined in these equivariant cohomology theory. Let $p : M \rightarrow pt$ be the constant map, we have equivariant Gysin maps:

$$\begin{aligned} p_!^G &: H^*(EG \times_G M) \rightarrow H^*(BG); \\ p_!^{G,K} &: K^*(EG \times_G M) \rightarrow K^*(BG); \\ p_!^{G,K_G} &: K_G^*(M) \rightarrow K_G^*(pt). \end{aligned}$$

Hence, the corresponding equivariant Chern numbers of M are defined to be:

$$\begin{aligned} c_\omega^G[M]_G &:= p_!^G(c_\omega^G(M)); \\ c_\omega^{G,K}[M]_G &:= p_!^{G,K}(c_\omega^{G,K}(M)); \\ c_\omega^{G,K_G}[M]_G &:= p_!^{G,K_G}(c_\omega^{G,K_G}(M)), \end{aligned}$$

respectively, where $\omega = (i_1, i_2, \dots, i_s)$ is a partition of $|\omega| = i_1 + i_2 + \dots + i_s$, and $c_\omega^G(M)$ (resp. $c_\omega^{G,K}(M), c_\omega^{G,K_G}(M)$) means the product $c_{i_1}^G(M)c_{i_2}^G(M) \cdots c_{i_s}^G(M)$ (resp. $c_{i_1}^{G,K}(M)c_{i_2}^{G,K}(M) \cdots c_{i_s}^{G,K}(M), c_{i_1}^{G,K_G}(M)c_{i_2}^{G,K_G}(M) \cdots c_{i_s}^{G,K_G}(M)$).

3.5. Inverse limits in equivariant cohomology theory of Borel type. Assume G is a compact Lie group. Let M be a compact unitary G -manifold and $EG \times_G M$ be the Borel construction of M . $EG \times_G M$ admits a filtration:

$$\cdots \subset EG(n) \times_G M \subset EG(n+1) \times_G M \subset \cdots \subset EG \times_G M,$$

where $EG(n)$ can be some compact smooth manifold. It is well-known that:

Proposition 3.6. *In ordinary cohomology theory, one has:*

$$\varprojlim H^i(EG(n) \times_G M) = H^i(EG \times_G M).$$

In complex K -theory, one also has:

$$\varprojlim K^i(EG(n) \times_G M) = K^i(EG \times_G M).$$

For a unitary G -vector bundle $EG \times_G E$ over $EG \times_G M$, let $EG \times_G D(E)$ be the disk bundle of $EG \times_G E$ and $EG \times_G S(E)$ be the sphere bundle of $EG \times_G E$. There is also a diagram:

$$\begin{array}{ccc} K^*(EG \times_G M) & \xrightarrow{ch} & H^{**}(EG \times_G M; \mathbb{Q}) \\ \psi_! \downarrow \cong & & \psi_* \downarrow \cong \\ K^*(EG \times_G D(E), EG \times_G S(E)) & \xrightarrow{ch} & H^{**}(EG \times_G D(E), EG \times_G S(E); \mathbb{Q}), \end{array}$$

where $\psi_!$ and ψ_* are Thom isomorphisms in K -theory and cohomology theory, respectively.

Proposition 3.7. *For any $x \in K(EG \times_G M)$, one has:*

$$ch(\psi_!(x)) = \psi_*(ch(x) \cdot Td_G^{-1}(E)),$$

where $Td_G(E) := Td(EG \times_G E) \in H^{**}(EG \times_G M; \mathbb{Q})$ is the total Todd class of the unitary vector bundle $EG \times_G E$.

Proof. By Proposition 3.6, we will use the finite approximation method to finish the desired proof. Let $i_n : EG(n) \times_G M \hookrightarrow EG \times_G M$ be the inclusion induced by $EG(n) \rightarrow EG$. For the Todd genus $Td_G(E)$, since $i_n^*(Td_G(E)) = i_n^*(Td(EG \times_G E)) = Td(EG(n) \times_G E)$, one has

$$\varprojlim Td(EG(n) \times_G E) = Td_G(E).$$

Take the inverse, one has that $\varprojlim Td^{-1}(EG(n) \times_G E) = Td_G^{-1}(E)$.

Let $x \in K^*(EG \times_G M)$ and $x_n = i_n^!(x) \in K^*(EG(n) \times_G M)$. One has the following commutative diagram:

$$\begin{array}{ccc} K^*(EG \times_G M) & \xrightarrow{i_n^!} & K^*(EG(n) \times_G M) \\ \psi_! \downarrow \cong & & \psi_{n!} \downarrow \cong \\ K^*(EG \times_G D(E), EG \times_G S(E)) & \xrightarrow{j_n^!} & K^*(EG(n) \times_G D(E), EG(n) \times_G S(E)), \end{array}$$

where j_n is the bundle map induced by i_n . The vector bundle $EG(n) \times_G E$ is the pull-back of $EG \times_G E$, so the Thom class $t(EG(n) \times_G E)$ of $EG(n) \times_G E$ satisfies $j_n^!(t(EG \times_G E)) = t(EG(n) \times_G E)$. It follows that $\varprojlim t(EG(n) \times_G E) = t(EG \times_G E)$.

By the definition of Thom isomorphism, we have that $\psi_{n!}(x_n) = \pi_n^!(x_n) \cdot t(EG(n) \times_G E)$ and $\psi_!(x) = \pi^!(x) \cdot t(EG \times_G E)$, where π_n (resp. π) denotes the projective map $EG(n) \times_G (D(E)/S(E)) \rightarrow EG(n) \times_G M$ (resp. $EG \times_G (D(E)/S(E)) \rightarrow EG \times_G M$). Since $\varprojlim \pi_n^!(x_n) = \pi^!(x)$ and $\varprojlim t(EG(n) \times_G E) = t(EG \times_G E)$, it follows that:

$$\begin{aligned} \psi_!(x) &= \pi^!(x) \cdot t(EG \times_G E) \\ &= \varprojlim \pi_n^!(x_n) \cdot \varprojlim t(EG(n) \times_G E) \\ &= \varprojlim (\pi_n^!(x_n) \cdot t(EG(n) \times_G E)) \\ &= \varprojlim \psi_{n!}(x_n). \end{aligned}$$

Similarly, one can show that

$$\psi_*(ch(x)Td_G^{-1}(E)) = \varprojlim \psi_{n*}(ch(x_n) \cdot Td^{-1}(EG(n) \times_G E)).$$

On the other hand, since $EG(n) \times_G M$ is compact, by the nonequivariant Riemann-Roch relation (see Proposition 3.3), we have:

$$ch(\psi_{n!}(x_n)) = \psi_{n*}(ch(x_n) \cdot Td^{-1}(EG(n) \times_G E)).$$

Take the inverse limit, one has:

$$\begin{aligned} ch(\psi_!(x)) &= ch(\varprojlim \psi_{n!}(x_n)) \\ &= \varprojlim ch(\psi_{n!}(x_n)) \\ &= \varprojlim \psi_{n*}(ch(x_n) \cdot Td^{-1}(EG(n) \times_G E)) \\ &= \psi_*(\varprojlim ch(x_n) \cdot \varprojlim Td^{-1}(EG(n) \times_G E)) \\ &= \psi_*(ch(x) \cdot Td_G^{-1}(E)) \end{aligned}$$

as desired. \square

3.6. An equivariant version of Riemann–Roch relation of Atiyah–Hirzebruch type. Let M and N be closed unitary G -manifolds and $f : M \rightarrow N$ be a G -map. Then f may induce equivariant Gysin maps

$$\begin{aligned} f_!^G &: H^{**}(EG \times_G M) \rightarrow H^{**}(EG \times_G N), \\ f_!^{G,K} &: K^*(EG \times_G M) \rightarrow K^*(EG \times_G N), \\ f_!^{G,K_G} &: K_G^*(M) \rightarrow K_G^*(N), \end{aligned}$$

in three equivariant cohomology theories $H^*(EG \times_G -)$, $K^*(EG \times_G -)$ and $K_G^*(-)$, respectively.

Since the Chern character is a natural transformation from K -theory to ordinary cohomology, one has the following diagram:

$$\begin{array}{ccc} K^*(EG \times_G M) & \xrightarrow{ch} & H^{**}(EG \times_G M; \mathbb{Q}) \\ f_!^{G,K} \downarrow & & f_!^G \downarrow \\ K^*(EG \times_G N) & \xrightarrow{ch} & H^{**}(EG \times_G N; \mathbb{Q}). \end{array}$$

Still, this diagram is NOT commutative. Similarly to the nonequivariant case, we have the following equivariant Riemann–Roch relation:

Theorem 3.8. *Let M and N be closed unitary G -manifolds with unitary stable tangent bundles, still denoted by TM and TN , respectively. Let $f : M \rightarrow N$ be a G -map. Then for any $x \in K^*(EG \times_G M)$,*

$$ch(f_!^{G,K}(x)) \cdot Td_G(TN) = f_!^G(ch(x) \cdot Td_G(TM)).$$

Proof. By the definition of equivariant Gysin map, choose an G -embedding $f \times e : M \hookrightarrow N \times V$ with normal bundle η , where V is a G -representation. Then one can obtain the following diagram:

$$\begin{array}{ccc}
K^*(EG \times_G M) & \xrightarrow{ch} & H^{**}(EG \times_G M; \mathbb{Q}) \\
\phi_1^K \downarrow \cong & & \phi_1^H \downarrow \cong \\
K^*(EG \times_G D(\eta), EG \times_G S(\eta)) & \xrightarrow{ch} & H^{**}(EG \times_G D(\eta), EG \times_G S(\eta); \mathbb{Q}) \\
\phi_2^K \downarrow & & \phi_2^H \downarrow \\
K^*(EG \times_G (N \times D(V)), EG \times_G (N \times S(V))) & \xrightarrow{ch} & H^{**}(EG \times_G (N \times D(V)), EG \times_G (N \times S(V)); \mathbb{Q}) \\
\phi_3^K \uparrow \cong & & \phi_3^H \uparrow \cong \\
K^*(EG \times_G N) & \xrightarrow{ch} & H^{**}(EG \times_G N; \mathbb{Q}).
\end{array}$$

Note that the middle square in the above diagram is commutative. For any $x \in K^*(EG \times_G M)$, by Proposition 3.7, we have the following relation:

$$ch(\phi_1^K(x)) = \phi_1^H(ch(x) \cdot Td_G^{-1}(\eta)).$$

Since $ch\phi_2^K = \phi_2^H ch$, we have:

$$ch(\phi_2^K \phi_1^K(x)) = \phi_2^H ch(\phi_1^K(x)) = \phi_2^H \phi_1^H(ch(x) \cdot Td_G^{-1}(\eta)).$$

Let $y = f_!^{G,K}(x) = (\phi_3^K)^{-1} \phi_2^K \phi_1^K(x)$, we see that $\phi_3^K(y) = \phi_2^K \phi_1^K(x)$ and

$$ch(\phi_3^K(y)) = \phi_3^H(ch(y) \cdot Td_G^{-1}(V)),$$

where $Td_G(V) = Td(EG \times_G V)$ is the equivariant total Todd class of the G -bundle $V \times N \rightarrow N$. Put these equalities together,

$$\begin{aligned}
ch(f_!^{G,K}(x)) \cdot Td_G^{-1}(V) &= ch(y) \cdot Td_G^{-1}(V) \\
&= (\phi_3^H)^{-1} ch(\phi_3^K(y)) \\
&= (\phi_3^H)^{-1} ch(\phi_2^K \phi_1^K(x)) \\
&= (\phi_3^H)^{-1} \phi_2^H \phi_1^H(ch(x) \cdot Td_G^{-1}(\eta)) \\
&= f_!^G(ch(x) \cdot Td_G^{-1}(\eta)),
\end{aligned}$$

and one has:

$$ch(f_!^{G,K}(x)) = f_!^G(ch(x) \cdot Td_G^{-1}(\eta)) \cdot Td_G(V).$$

On the other hand, we see that $TM \oplus \eta = (f \times e)^*T(N \times V) \cong f^*(TN \oplus V)$, where $f^*(TN \oplus V)$ is the pull-back of the bundle $TN \oplus V \rightarrow N$ via $f : M \rightarrow N$. It follows that:

$$Td_G(\eta) \cdot Td_G(TM) = f^*(Td_G(TN) \cdot Td_G(V)).$$

Thus,

$$\begin{aligned}
f_!^G(ch(x) \cdot Td_G(TM)) &= f_!^G(ch(x) \cdot Td_G^{-1}(\eta) \cdot f^*(Td_G(TN) \cdot Td_G(V))) \\
&= f_!^G(ch(x) \cdot Td_G^{-1}(\eta)) \cdot Td_G(V) \cdot Td_G(TN) \text{ (by Proposition 4.4)} \\
&= ch(f_!^{G,K}(x)) \cdot Td_G(TN)
\end{aligned}$$

as desired. \square

There is a natural transformation from $K_G(-)$ to $K(EG \times_G -)$:

$$\alpha : K_G(X) \longrightarrow K(EG \times_G X)$$

$$E \mapsto EG \times_G E,$$

for any G -space X and unitary G -vector bundle E over X . Then we define the equivariant Chern character as follows:

Definition 3.9. For any G -space X , the *equivariant Chern character* ch_G is defined by the composition of α and ch :

$$ch_G := ch \circ \alpha : K_G(X) \longrightarrow K(EG \times_G X) \longrightarrow H^{**}(EG \times_G X; \mathbb{Q}).$$

By choosing the Thom classes, for any G -map $f : M \rightarrow N$ between two closed unitary G -manifolds, one has the following commutative diagram:

$$\begin{array}{ccc}
K_G(M) & \xrightarrow{\alpha} & K(EG \times_G M) \\
f_!^{G,K_G} \downarrow & & f_!^{G,K} \downarrow \\
K_G(N) & \xrightarrow{\alpha} & K(EG \times_G N).
\end{array}$$

Proposition 3.10. *Let M and N be closed unitary G -manifolds with unitary stable tangent bundles which are still denoted by TM and TN . Let $f : M \rightarrow N$ be a G -map. Then for any $x \in K_G^*(M)$,*

$$ch_G(f_!^{G,K_G}(x)) \cdot Td_G(TN) = f_!^G(ch_G(x) \cdot Td_G(TM)).$$

Proof. By Theorem 3.8, for $\alpha(x) \in K^*(EG \times_G M)$, one has:

$$ch(f_!^{G,K}(\alpha(x))) \cdot Td_G(TN) = f_!^G(ch(\alpha(x)) \cdot Td_G(TM)).$$

Since $\alpha f_!^{G,K_G} = f_!^{G,K} \alpha$, it follows that

$$\begin{aligned}
ch_G(f_!^{G,K_G}(x)) \cdot Td_G(TN) &= ch(\alpha f_!^{G,K_G}(x)) \cdot Td_G(TN) \\
&= ch(f_!^{G,K} \alpha(x)) \cdot Td_G(TN) \\
&= f_!^G(ch(\alpha(x)) \cdot Td_G(TM)) \\
&= f_!^G(ch_G(x) \cdot Td_G(TM))
\end{aligned}$$

as desired. \square

3.7. Equivalence of integral equivariant cohomology Chern numbers and equivariant K-theoretic Chern numbers. Now we are going to prove:

Theorem 3.11. *Let M be a unitary T^k -manifold with stable almost complex tangent bundle $TM \in K_{T^k}(M)$. Then all integral equivariant cohomology Chern numbers of M vanish if and only if all equivariant K-theoretic Chern numbers of M vanish.*

Proof. First, assume that all integral equivariant cohomology Chern numbers of M vanish. Then for any equivariant K-theoretic Chern number

$$c_{\omega}^{T^k, K_{T^k}}[M]_{T^k} = p_!^{T^k, K_{T^k}}(c_{\omega}^{T^k, K_{T^k}}(M))$$

where ω is a partition, one has by Proposition 3.10 that

$$\begin{aligned} ch_{T^k}(c_{\omega}^{T^k, K_{T^k}}[M]_{T^k}) &= ch_{T^k}(p_!^{T^k, K_{T^k}}(c_{\omega}^{T^k, K_{T^k}}(M))) \\ &= p_!^{T^k}(ch_{T^k}(c_{\omega}^{T^k, K_{T^k}}(M)) \cdot Td_{T^k}(TM)) \end{aligned}$$

Since $ch_{T^k}(c_{\omega}^{T^k, K_{T^k}}(M)) \cdot Td_{T^k}(TM)$ is a combination of equivariant cohomology Chern classes, it follows that $ch_{T^k}(c_{\omega}^{T^k, K_{T^k}}[M]_{T^k})$ can be represented by a combination of equivariant cohomology Chern numbers. By assumption, one has

$$ch_{T^k}(c_{\omega}^{T^k, K_{T^k}}[M]_{T^k}) = 0.$$

On the other hand, $ch_{T^k} : K_{T^k}(pt) \rightarrow H_{T^k}^{**}(pt; \mathbb{Q})$ is injective and so the equivariant K-theoretic Chern number $c_{\omega}^{T^k, K_{T^k}}[M]_{T^k} = 0$.

Second, assume all the equivariant K-theoretic Chern numbers vanish. For any integral equivariant cohomology Chern number $c_{\omega}^{T^k}[M]_{T^k}$, consider the equivariant K-theoretic Chern class $c_{\omega}^{T^k, K_{T^k}}(M)$. Then one has

$$ch_{T^k}(c_{\omega}^{T^k, K_{T^k}}(M)) = c_{\omega}^{T^k}(M) + \text{terms of higher degree in } H_{T^k}^{**}(M; \mathbb{Q}),$$

and so

$$\begin{aligned} 0 &= ch_{T^k}(c_{\omega}^{T^k, K_{T^k}}[M]_{T^k}) \\ &= ch_{T^k}(p_!^{T^k, K_{T^k}}(c_{\omega}^{T^k, K_{T^k}}(M))) \\ &= p_!^{T^k}(ch_{T^k}(c_{\omega}^{T^k, K_{T^k}}(M)) \cdot Td_{T^k}(TM)) \\ &= p_!^{T^k}(c_{\omega}^{T^k}(M) + \text{terms of higher degree}) \\ &= c_{\omega}^{T^k}[M]_{T^k} + \text{terms of higher degree in } H_{T^k}^{**}(pt; \mathbb{Q}). \end{aligned}$$

It follows that $c_{\omega}^{T^k}[M]_{T^k} = 0$ in $H_{T^k}^{**}(pt; \mathbb{Q}) = H^{**}(BT^k; \mathbb{Q})$. This implies that $c_{\omega}^{T^k}[M]_{T^k} = 0$ in $H^*(BT^k)$ since Ω_*^{U, T^k} and $H^*(BT^k)$ have no torsion (cf [19]). Thus we have proved that all integral equivariant cohomology Chern numbers of M vanish. \square

4. APPENDIX

In this section, we will review some standard facts on (equivariant) Gysin maps (cf [2, 9, 15, 16]).

4.1. Definition of Gysin maps. Let M^m and N^n be two closed oriented smooth manifolds and $f : M^m \rightarrow N^n$ be a smooth map. Then the Gysin map $f_! : H^*(M^m) \rightarrow H^{*-(m-n)}(N^n)$ can be defined as follows:

First, we embed M^m into \mathbb{R}^{m+p} for sufficiently large p and denote this embedding by $j : M^m \hookrightarrow \mathbb{R}^{m+p}$. Consider the embedding

$$f \times j : M^m \rightarrow N^n \times \mathbb{R}^{m+p};$$

and denote the normal bundle of M^m by η_M . Since M^m is compact, it is safe to assume that the image of the embedding $f \times j$ stays in the interior of $N^n \times D^{m+p}$, where D^{m+p} denotes the $(m+p)$ -disk.

Second, there are three homomorphisms:

$$\begin{aligned} \phi_1 : H^*(M^m) &\rightarrow \tilde{H}^{*+n+p}(D(\eta_M), S(\eta_M)), & (\text{Thom isomorphism}) \\ \phi_2 : \tilde{H}^{*+n+p}(D(\eta_M), S(\eta_M)) &\rightarrow \tilde{H}^{*+n+p}(N^n \times D^{m+p}, N^n \times S^{m+p-1}), & (\text{collapsing map}) \\ \phi_3 : H^{*+n-m}(N^n) &\rightarrow \tilde{H}^{*+n+p}(N^n \times D^{m+p}, N^n \times S^{m+p-1}). & (\text{Thom isomorphism}) \end{aligned}$$

Third, the Gysin map is defined by:

$$f_! := \phi_3^{-1} \circ \phi_2 \circ \phi_1 : H^*(M^m) \rightarrow H^{*-(m-n)}(N^n).$$

Remark 8. In fact, $f_!$ depends on the homotopy class of f . Namely, if f and g are homotopic, then the embeddings $f \times j$ and $g \times j$ are isotopic to each other and it follows that $f_! = g_!$. If $f : M^m \rightarrow N^n$ is continuous, by smoothing approximation theorem, we can also define the Gysin map $f_! : H^*(M^m) \rightarrow H^{*+n-m}(N^n)$.

More generally, let E^* be a generalized multiplicative cohomology theory, say complex K -theory K^* and complex cobordism theory MU^* , then Thom isomorphism theory still holds. Therefore, we can also define the Gysin map

$$f_! : E^*(M^m) \rightarrow E^{*+n-m}(N^n).$$

In this paper, let $f_!^K$ denote the Gysin map in complex K -theory

$$f_!^K : K^*(M^m) \rightarrow K^{*+n-m}(N^n);$$

and let $f_!^{MU}$ denote the Gysin map in complex cobordism theory

$$f_!^{MU} : MU^*(M^m) \rightarrow MU^{*+n-m}(N^n).$$

4.2. Properties of Gysin maps. Let M and N be two closed oriented smooth manifolds and $f : M \rightarrow N$ be a smooth map. Now let us list some important properties of Gysin maps in ordinary cohomology.

Proposition 4.1 (cf [9]). *Let $f_! : H^*(M) \rightarrow H^*(N)$ be the Gysin map induced by f .*

(1) *There is the following commutative diagram:*

$$\begin{array}{ccc} H^*(M) & \xrightarrow{f_!} & H^*(N) \\ \cap[M] \downarrow & & \cap[N] \downarrow \\ H_*(M) & \xrightarrow{f_*} & H_*(N), \end{array}$$

where the vertical arrows are isomorphisms induced by Poincaré duality.

- (2) In particular, let $p : M \longrightarrow pt$ be the constant map, then for each $x \in H^*(M)$, one has:

$$p_!(x) = \langle x, [M] \rangle,$$

where $[M]$ is the fundamental homology class of M .

Proposition 4.2 (cf [9]). Let $f_! : H^*(M) \longrightarrow H^*(N)$ be the Gysin map induced by f .

- (1) For $\alpha \in H^*(N)$ and $\beta \in H^*(M)$, one has $f_!(f^*(\alpha) \cup \beta) = \alpha \cup f_!(\beta)$.
- (2) If $\dim M > \dim N$, then $f_!(1) = 0$.
- (3) Let L be a closed oriented manifold and let $g : N \longrightarrow L$ be a continuous map. Then $(gf)_! = g_!f_! : H^*(M) \longrightarrow H^*(L)$.

Proposition 4.3 (cf [9]). If $\pi : E \longrightarrow N$ is a fiber bundle of manifold over N , then the pull-back $\pi' : \tilde{f}^*(E) \longrightarrow M$ via f

$$\begin{array}{ccc} \tilde{f}^*(E) & \xrightarrow{\tilde{f}} & E \\ \pi' \downarrow & & \downarrow \pi \\ M & \xrightarrow{f} & N \end{array}$$

is also a fiber bundle of manifold. Moreover, this pull-back square induces that $f^*\pi_! = \pi'_!\tilde{f}^*$.

Remark 9. For complex K -theory $K^*(-)$ and complex cobordism theory $MU^*(-)$, Proposition 4.2 and Proposition 4.3 still hold.

Remark 10. According to [27], for a complex orientation of the map $f : M \longrightarrow N$

$$M \hookrightarrow E \longrightarrow N,$$

where E is a complex vector bundle over N . Denote η_M by the normal bundle of the inclusion $M \hookrightarrow E$, One can also define a Gysin homomorphism $f_!^E : H^*(M) \longrightarrow H^*(N)$ to be

$$f_!^E : H^*(N) \cong H^*(D(\eta_M)/S(\eta_M)) \longrightarrow H^*(D(E)/S(E)) \cong H^*(N).$$

When $E = N \times \mathbb{C}^{\mathcal{N}}$ for some positive integer \mathcal{N} , this homomorphism is just the Gysin map defined above. In fact, we see easily that $f_!^E$ doesn't depend on the vector bundle E . More precisely, $f_! = f_!^E$ for any complex orientation of the map $f : M \longrightarrow N$.

4.3. Equivariant Gysin map. Let $h_G(-)$ be an equivariant multiplicative cohomology theory, say $H^*(EG \times_G -)$, $K^*(EG \times_G -)$, $K_G^*(-)$ and $MU^*(EG \times_G -)$. In $h_G(-)$, equivariant Gysin maps are still well-defined. Namely, let M and N be two closed unitary G -manifolds and $f : M \longrightarrow N$ be an equivariant G -map. Then the equivariant Gysin map

$$f_!^G : h_G^*(M) \longrightarrow h_G^*(N)$$

is also defined in a similar way to nonequivariant case. There are the following three steps:

Step 1. Embed M equivariantly into a G -vector space V with embedding $e : M \hookrightarrow V$.

Step 2. Consider the embedding $f \times e : M \longrightarrow N \times V$ with equivariant normal bundle η and we then obtain three homomorphisms:

$$\begin{aligned} \phi_1 : h_G(M) &\longrightarrow \widetilde{h}_G(D(\eta), S(\eta)), & (\text{Thom isomorphism}) \\ \phi_2 : \widetilde{h}_G(D(\eta), S(\eta)) &\longrightarrow \widetilde{h}_G(N \times D(V), N \times S(V)), & (\text{collapsing map}) \\ \phi_3 : h_G(N) &\longrightarrow h_G(N \times D(V), N \times S(V)). & (\text{Thom isomorphism}) \end{aligned}$$

Step 3: Define $f_!^G$ by

$$f_!^G := \phi_3^{-1} \circ \phi_2 \circ \phi_1 : h_G(M) \longrightarrow h_G(N).$$

In particular, when N is a point, the constant map $p : M \longrightarrow pt$ induces the equivariant Gysin map:

$$p_!^G : h_G(M) \longrightarrow h_G(pt).$$

4.4. Properties of equivariant Gysin maps. Now let us list the following properties of equivariant Gysin maps, which are similar to the nonequivariant case.

Proposition 4.4 (cf [2]).

- (1) *Let M and N be two closed unitary G -manifolds and $f : M \longrightarrow N$ be an equivariant G -map. For $\alpha \in h_G^*(N)$ and $\beta \in h_G^*(M)$, one has*

$$f_!^G(f^*(\alpha) \cup \beta) = \alpha \cup f_!^G(\beta).$$

- (2) *Let $f : M_1 \longrightarrow M_2$ and $g : M_2 \longrightarrow M_3$ be two equivariant G -maps. Then one has*

$$(gf)_!^G = g_!^G f_!^G : h_G^*(M_1) \longrightarrow h_G^*(M_3).$$

Let M be a closed unitary G -manifold of dimension m and $p : M \longrightarrow pt$ be the constant map, where G is a torus. We note that the equivariant Gysin map of p is defined to be $p_!^G : H_G^*(M) \longrightarrow H_G^{*-m}(pt)$ in the above way, which agrees with the ordinary Gysin map $\pi_!$ of $\pi : EG \times_G M \longrightarrow BG$ given by the complex orientation

$$EG \times_G M \hookrightarrow EG \times_G V \longrightarrow BG$$

of the proper map $\pi : EG \times_G M \longrightarrow BG$, where V is a G -representation space. Namely

$$\begin{array}{ccc} H^*(EG \times_G M) & \xlongequal{\quad} & H_G^*(M) \\ \pi_! \downarrow & & p_!^G \downarrow \\ H^{*-m}(BG) & \xlongequal{\quad} & H_G^{*-m}(pt). \end{array}$$

Now let us look at the pull-back from the fiber bundle $\pi : EG \times_G M \longrightarrow BG$. Let X be a closed unitary manifold and $f : X \longrightarrow BG$ be a smooth map. Then one has the following pull-back square

$$(4.1) \quad \begin{array}{ccc} \widetilde{f}^*(EG \times_G M) & \xrightarrow{\quad \widetilde{f} \quad} & EG \times_G M \\ \pi' \downarrow & & \pi \downarrow \\ X & \xrightarrow{\quad f \quad} & BG \end{array}$$

such that $\pi' : \widetilde{f}^*(EG \times_G M) \longrightarrow X$ is a fiber bundle of manifold by Proposition 4.3. As noted in Subsection 2.4 of this paper, $\widetilde{f}^*(EG \times_G M)$ is a unitary manifold of dimension

$\dim M + \dim X$. The Gysin map $\pi'_! : H^*(\tilde{f}^*(EG \times_G M)) \rightarrow H^{*-m}(X)$ can also be given by using the induced complex orientation

$$\tilde{f}^*(EG \times_G M) \hookrightarrow \tilde{f}^*(EG \times_G V) \rightarrow X$$

of the map $\pi' : \tilde{f}^*(EG \times_G M) \rightarrow X$. As noted in Remark 10, $\pi'_!$ is just the ordinary Gysin map induced by π' . Applying Proposition 4.3 and the finite approximation method gives the following result.

Proposition 4.5. *The pull-back square (4.1) yields the following commutative diagram:*

$$\begin{array}{ccc} H^*(EG \times_G M) & \xrightarrow{\tilde{f}^*} & H^*(\tilde{f}^*(EG \times_G M)) \\ \pi'_! \downarrow & & \downarrow \pi'_! \\ H^{*-m}(BG) & \xrightarrow{f^*} & H^{*-m}(X). \end{array}$$

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